Statistical Analysis of Static Shape Control in Space Structures

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The article addresses the problem of efficient analysis of the statistics of initial and corrected shape distortions in space structures. Two approaches for improving efficiency are considered. One is an adjoint technique for calculating distortion shapes; the second is a modal expansion of distortion shapes in terms of pseudo-vibration modes. The two techniques are applied to the problem of optimizing actuator locations on a 55 m radiometer antenna. The adjoint analysis technique is used with a discrete-variable optimization method. The modal approximation technique is coupled with a standard conjugate-gradient continuous optimization method. The agreement between the two sets of results is good, validating both the approximate analysis and optimality of the results.

Introduction

MANUFACTURING errors are recognized as a major cause of shape distortion in large space structures.[1] These errors as well as many other sources of distortion such as variations in the coefficient of thermal expansion are random in nature, and a statistical analysis of structural response must be performed to estimate the resulting shape distortion. This statistical analysis can become computationally costly when the structure is a large truss structure with many thousands of members. Reference 1 presented a modal expansion technique that under some restrictive conditions resulted in a simple and inexpensive estimate for the statistics of the distortion. Reference 2 dealt with the problem by Monte-Carlo simulation.

When the expected shape distortion is deemed unacceptable, it is often corrected by static shape actuators,[3] which typically control the length of the truss elements. The location of these actuators needs to be designed to maximize their effectiveness. Reference 4 presented a discrete-variable algorithm for obtaining near optimal actuator locations and applied it to the problem of correcting a prescribed deterministic distortion. Reference 5 presented a statistical analysis of corrected distortion and used it to obtain optimal actuator locations for a beam truss. However, the methods employed in Ref. 5 are not applicable to the solution of optimal actuator location for three-dimensional space trusses. For the more general problem the expense of the statistical analysis is substantial, and the problem of choosing actuator locations among a discrete set of available locations requires a large number of analyses.

The objective of the present article is to consider several alternatives for improving the efficiency of the statistical analysis of the corrected distortion. Two of these alternatives are then applied to the problem of optimizing actuator locations on an antenna truss structure.

Problem Definition

The structure is assumed to have a set of n_s sensors that measure n_s components of the distortion field, and a set of n actuators used to correct the distortion. Denoting the vector of sensor-measured components of the distortion as ψ, we assume that the actuators seek to minimize a quadratic measure of the distortion $\psi_{\rm rms}$ defined as

$$\psi^2_{\rm rms} = \psi^T B \psi \tag{1}$$

where B is a positive semidefinite matrix (for unweighted rms B is equal to $[(1/n_s)I]$. We assume that the behavior of the structure and actuators is linear, so that if a unit action by the ith actuator produces a displacement vector u_i then the corrected shape vector δ is given as

$$\delta = \psi + \sum_{i=1}^{n} u_i \theta_i = \psi + U\theta \tag{2}$$

where θ is the vector of actuator outputs (with components θ_i) and U is a matrix with columns u_i. The corrected measure of distortion is $\delta_{\rm rms}$

$$\delta^2_{\rm rms} = \delta^T B \delta \tag{3}$$

The optimum vector θ that minimizes $\delta_{\rm rms}$ is known[4] to be the solution of the system

$$A\theta = r \tag{4}$$

where

$$A = U^T B U$$
$$r = -U^T B \psi \tag{5}$$

and then the corrected shape δ is

$$\delta = (I - UA^{-1}U^T B)\psi = G\psi \tag{6}$$

and

$$\delta^2_{\rm rms} = \psi^T G^T B G \psi \tag{7}$$

An alternate expression for $\delta_{\rm rms}$ is

$$\delta_{\rm rms} = \psi^2_{\rm rms} - \psi^2_{\rm crms} \tag{8}$$

where

$$\psi^2_{\rm crms} = \psi^T B U A^{-1} U^T B \psi \tag{9}$$

The effectiveness of the actuators is measured by the distortion ratio g, defined as

$$g^2 = \frac{E^{(\delta^2_{\rm rms})}}{E^{(\psi^2_{\rm rms})}} = 1 - \frac{E^{(\psi^2_{\rm crms})}}{E^{(\psi^2_{\rm rms})}} \tag{10}$$

where E denotes the expected value. In the present work we

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define the optimum location of the actuators to be the one that gives the minimum of g^2.

Statistical Analysis

Let the distortion field be due to a set of errors or disturbances in a structure characterized by their amplitude parameters ϵ_i, $i = 1,2...N$. The behavior of the structure is assumed to be linear, so that the total distortion due to the N errors, ψ, is given as

$$\psi = \sum_{i=1}^{N} \epsilon_i \psi_i = \Psi e \tag{11}$$

where ψ_i is the shape distortion due to a unit ϵ_i, Ψ is a matrix with ψ_i as its ith column, and e is a vector of ϵ_i.

We assume that the ϵ_i are uncorrelated, zero-mean normally distributed random variables with a standard deviation σ_ϵ, so that the covariance matrix of e is $\sigma_\epsilon^2 I$. From Eq. (11) the covariance matrix of Ψ

$$C_{\psi\psi} = \sigma_\epsilon^2 \Psi \Psi^T \tag{12}$$

To calculate the statistics of ψ_{rms}^2 we rewrite Eq. (1) as

$$\psi_{\text{rms}}^2 = \psi^T B \psi = \bar{\psi}^T \bar{\psi} \tag{13}$$

where

$$\bar{\psi} = L^T \psi \tag{14}$$

and L is the Cholesky factor of the weighting matrix B (*i.e.* $B = LL^T$). Then the covariance matrix $\bar{C}_{\psi\psi}$ of $\bar{\psi}$ is given as

$$\bar{C}_{\psi\psi} = L^T C_{\psi\psi} L \tag{15}$$

and the expected value of ψ_{rms}^2 is

$$E(\psi_{\text{rms}}^2) = \sum_i \sigma^2(\bar{\psi}_i) \tag{16}$$

where $\sigma^2(\bar{\psi}_i)$, the variances of the components of $\bar{\psi}$, are the diagonal elements of $\bar{C}_{\psi\psi}$. From Ref. 6 we also have that Eq. (13) implies that the variance of ψ_{rms}^2 is

$$\sigma^2(\psi_{\text{rms}}^2) = 2\sum_i \sum_j (\bar{C}_{\psi\psi})_{ij}^2 \tag{17}$$

The statistics of the corrected shape are obtained in a similar manner. From Eq. (6) the covariance of δ is

$$C_{\delta\delta} = G C_{\psi\psi} G^T \tag{18}$$

By analogy with the uncorrected case we define

$$\bar{\delta} = L^T \delta \tag{19}$$

which has a covariance matrix $\bar{C}_{\delta\delta}$ given as

$$\bar{C}_{\delta\delta} = L^T C_{\delta\delta} L \tag{20}$$

and then

$$E(\sigma_{\text{rms}}^2) = \sum_i \sigma^2(\bar{\delta}_i) \tag{21}$$

and

$$\sigma^2(\delta_{\text{rms}}^2) = 2\sum_i \sum_j (\bar{C}_{\delta\delta})_{ij}^2 \tag{22}$$

Another way to calculate the expected value of δ_{rms}^2, which can be cheaper computationally, is based on Eq. (8), which can be written as

$$\delta_{\text{rms}}^2 = \psi_{\text{rms}}^2 - W^T W \tag{23}$$

where, from Eq. (9)

$$W = L_1^T U^T B \psi \tag{24}$$

and L_1 is the Cholesky factor of A^{-1}. The covariance matrix of W is then given as

$$C_{WW} = L_1^T U^T B C_{\psi\psi} B U L_1 \tag{25}$$

and

$$E(\delta_{\text{rms}}^2) = \sum_i [\sigma^2(\bar{\psi}_i) - \sigma^2(w_i)] \tag{26}$$

where $\sigma^2(w_i)$, the variances of the componets of W, are the diagonal terms of C_{WW}.

Efficient Calculation of ψ_i

The shape distortion due to a unit ϵ_i, denoted ψ_i, is typically calculated by finite element analysis from

$$K \bar{\psi}_i = F_i \tag{27}$$

where K is the stiffness matrix, $\bar{\psi}_i$ is an expanded ψ_i (to include nonsensed components; from now on we will follow the convention that a barred quantity has all the degrees of freedom of the finite element model), and F_i is a force vector representing the effects of the ith error source. The distortion is measured from the best-fit paraboloid to the distorted surface. If that paraboloid is constrained to have the same focal length as the undeformed surface, a least-square fit can be shown to be equivalent to require $\bar{\psi}_i$ to be orthogonal to the rigid body modes with respect to B, that is

$$R^T B \bar{\psi}_i = 0 \tag{28}$$

where R is a matrix of six rigid-body-motion vectors. Due to the rigid-body-motion freedom K is singular, and it is convenient to add a set of rigid-body-motion constraints, form a constrained stiffness matrix $\tilde{K}$ and solve

$$\tilde{K} \tilde{\psi}_i = F_i \tag{29}$$

The vector $\tilde{\psi}_i$ is different from $\bar{\psi}_i$ only in rigid body motion

$$\tilde{\psi}_i = \bar{\psi}_i + R\alpha \tag{30}$$

where α is a vector of rigid-body-motion amplitudes. Using Eq. (28) to find α we get

$$\bar{\psi}_i = P \tilde{\psi}_i \tag{31}$$

where the projection matrix P is given as

$$P = I - R(R^T B R)^{-1} R^T B \tag{32}$$

Note that P is typically not calculated because we will need only its product by a vector. Such products are more efficiently obtained by multiplying $\tilde{\psi}_i$ successively by B, R^T, etc. After $\bar{\psi}_i$ is calculated, the sensed components are extracted from it in a process that can be written as

$$\psi_i = H \bar{\psi}_i \tag{33}$$

Equations (29), (31) and (33) can be summarized as

$$\psi_i = H P \tilde{K}^{-1} F_i \tag{34}$$

Equation (34) immediately leads to an adjoint approach for an efficient calculation of ψ_i. Indeed, defining the adjoint matrix Λ as the solution of

$$\tilde{K} \Lambda = P^T H^T \tag{35}$$

Then Eq. (34) becomes

$$\psi_i = \Lambda^T F_i \tag{36}$$

The solution of Eq. (35) for Λ is equivalent to solving the response of the strucutre to n_s load vectors. Thus, no matter how many error sources we have, we need to solve the response of the structure to only a small number of loads.

Modal Approximation

Another approach to efficient calculation of the statistical properties of the distortion was proposed in Ref. 1. It is based on a modal approximation of ψ_i in terms of the "vibration modes" obtained by using B as the mass matrix. That is the modes are obtained as the solution of the eigenvalue problem

$$(K - \omega^2 \bar{B})\bar{\phi} = 0 \tag{37}$$

where $\bar{B}$ is an expansion of B to the full set of degrees of freedom of the structure. The ith eigenvalue of Eq. (37) is denoted ω_i^2 and the ith eigenvector, compressed back to include only the sensed components, is denoted ϕ_i and is normalized so that

$$\phi_i^T B \phi_i = \bar{\phi}_i{}^T \bar{B} \bar{\phi}_i = 1 \tag{38}$$

Even though K can be a very large matrix, Eq. (37) has only a maximum of n_s eigenvalues. We now use the eigenvectors ϕ_i as a basis for approximating the distortion. That is, each unit distortion vector $\bar{\psi}_i$ is approximated as

$$\bar{\psi}_i = \sum_{j=1}^{m} q_{ij} \bar{\phi}_j \tag{39}$$

where q_{ij} are the modal coordinates of $\bar{\phi}_i$. If m is equal to n_s, the approximation is complete. That is, even though $\bar{\psi}_i$ is not exactly equal to the right-hand-side of Eq. (39), the difference does not contribute anything to $\psi_{\rm rms}$.

The modal approximation requires the solution of a costly eigenvalue problem, but it trivializes the calculation of the ψ_i. Because the vibration modes satisfy

$$\bar{\phi}_i{}^T K \bar{\phi}_j = \delta_{ij} \omega_i^2 \tag{40}$$

we have by using Eqs. (27) and (39)

$$q_{ij} = \bar{\phi}_j{}^T F_i / \omega_j^2 \tag{41}$$

The statistical analysis in terms of modal coordinates is identical to the statistical analysis in terms of physical coordinates except that the matrix B is replaced by the identity matrix, and the matrix $\bar{\psi}$ replaced by the matrix Q (with components q_{ij}). This can be easily verified, for example,

$$\psi_i{}^T B \psi_i = \left(\sum_j q_{ij} \bar{\phi}_j\right)^T B \sum_k q_{ik} \bar{\phi}_k = \sum_j q_{ij}^2 \tag{42}$$

in view of the orthonormality of the ϕ_j. The right-hand-side of Eq. (42) is $\psi_i^T \psi_i$ in modal coordinates. The covariance matrix $C_{\psi\psi}$ in modal coordinates is given as [see Eq. (10)]

$$(C_{\psi\psi})_{k\ell} = \sigma_\epsilon^2 \sum_{i=1}^{N} q_{ik} q_{i\ell} = \sigma_\epsilon^2 \sum_{i=1}^{N} (\bar{\phi}_k^T F_i)(\bar{\phi}_\ell^T F_i)/\omega_k^2 \omega_\ell^2 \tag{43}$$

Truss Structures

For truss structures, and manufacturing errors manifested in member-length variations, it is possible to get simpler expressions. Denoting by p_{ij} the load in member i due to the jth mode, the orthogonality of the modes with respect to the stiffness matrix can be written as

$$\bar{\phi}_k{}^T K \bar{\phi}_\ell = \sum_{i=1}^{N} \ell_i p_{ik} p_{i\ell}/(EA)_i = \omega_k^2 \delta_{k\ell} \tag{44}$$

where $(EA)_i$ and ℓ_i are the rigidity and the length of the ith member, respectively. If we assume that a unit error ($\epsilon_i = 1$) corresponds to a change $\Delta\ell_i$ in the ith member, then the correspond vector F_i is a pair of colinear forces of magnitude $(EA)_i \Delta\ell_i/\ell_i$ aligned with member i, and so

$$\bar{\phi}_j{}^T F_i = \Delta\ell_i p_{ij} \tag{45}$$

Hedgepeth[1] achieved spectacular simplification of Eq. (43) by assuming that

$$\Delta\ell_i^2 \, (EA)_i/\ell_i = C \tag{46}$$

is the same for all members. Then Eq. (43) becomes

$$(C_{\psi\psi})_{k\ell} = \sigma_\epsilon^2 \sum_{i=1}^{N} p_{ik} p_{i\ell} \Delta\ell_i^2/\omega_k^2 \omega_\ell^2 = C\sigma_\epsilon^2 \delta_{k\ell}/\omega_k^2 \tag{47}$$

and from Eqs. (16) and (17) we get

$$E(\psi_{\rm rms}^2) = C\sigma_\epsilon^2 \sum_k \frac{1}{\omega_k^2} \tag{48}$$

$$\delta^2(\psi_{\rm rms}^2) = 2\, C^2 \sigma_\epsilon^4 \sum_k \frac{1}{\omega_k^4} \tag{49}$$

Optimum Actuator Locations on Antenna Structure

The efficiency of the statistical analysis is particularly important in the context of design where the analysis has to be repeated many times. Here we consider an example of finding optimum actuator locations to correct manufacturing errors for the truss structure of a 55-m radiometer antenna shown in Fig. 1. The only errors considered were variations in the lengths of the truss members. The reflector is built up from tetrahedral truss modules, and consists of 420 members connected at 109 joints. The upper surface of the truss supports the reflecting surface, and the r.m.s. of the vertical motion of this surface was used as a measure of the error. This corresponds to the vector ψ consisting of the vertical displacements of the 61 joints on the upper surface, and the B matrix equal to (1/61) I.

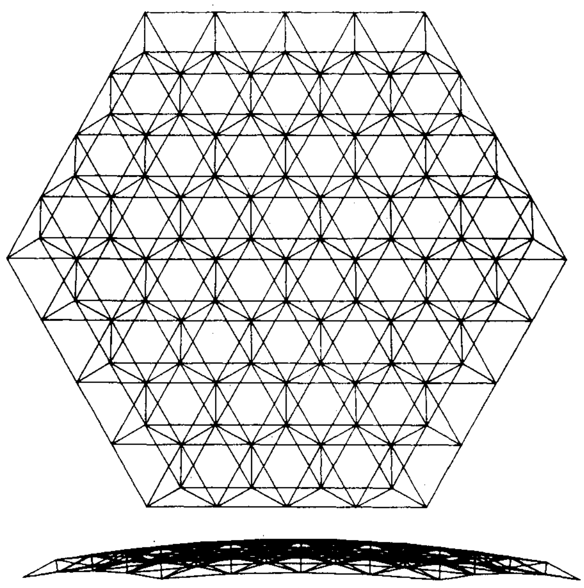

Fig. 1 Side and top views of tetrahedral truss antenna reflector.

The actuators were assumed to be embedded in a subset of the 156 upper-surface elements, and to control the length of the elements in this subset. The problem of optimum location is, therefore, the problem of choosing n of the 156 elements.

A two-pronged approach to the optimization was undertaken. First, the discrete-variable optimization method Exhaustive Single-Point Substitution[4] (ESPS) was employed with an exact statistical analysis employing the adjoint approach of calculating ψ_i. The ESPS method is based on replacing one actuator at a time in a trial configuration, and can converge to a nonoptimal solution. To check that this did not happen, a continuous optimization approach was applied to an approximate problem.

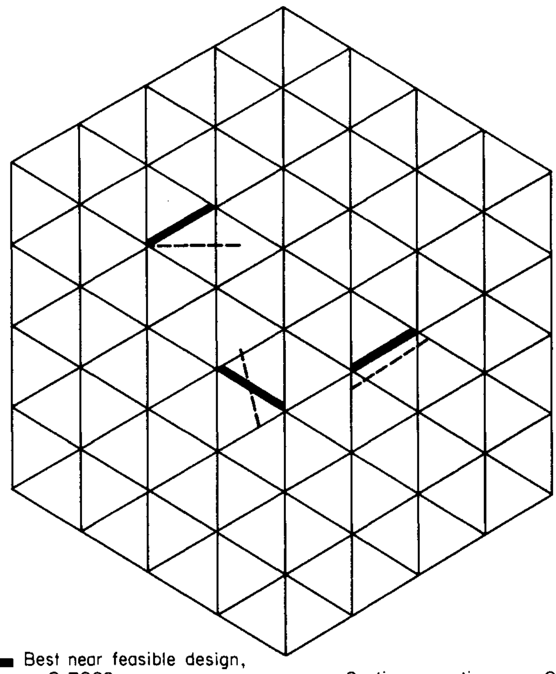

Fig. 2 Optimum locations of three actuators by continuous approximate optimization (values of g based on modal approximation with 34 modes).

Two assumptions are made to obtain the approximate problem: 1) the actuators can be located anywhere on the upper surface of the antenna; and 2) the modal approximation is used. The design variables for the approximate problem were two coordinates defining the position of the actuator and an orientation angle. The actuator was assumed to be a variable length element with the same initial length and rigidity as the upper-surface elements.

To calculate the displacment field u_i associated with the ith actuator, a pair of colinear forces F_i required for a unit elongation were assumed to act at the two ends of the actuator element. The vibration modes were interpolated by a two-dimensional spline to the two ends of the element to calculate the modal extension of the actuator element. This modal extension was then multiplied by the force F_i in that element required to simulate the required extension of the actuator to obtain $\bar{\phi}_j{}^T F_i$ in Eq. (41). Thus the modal approximation allowed us to estimate the effect of actuator elements in positions and orientations not available on the actual truss. This, in turn, converts the optimization problem to a continuous-variable problem that can be used by standard optimization techniques. We used a standard conjugate gradient algorithm as coded in the IMSL routine ZXCGR. The structural anlaysis was performed with the EAL finite element program.[7]

Results and Discussion

The continuous and discrete approaches were compared first for three actuators using 34 modes in the modal approximation. The results are summarized in terms of the distortion ratio g of Eq. (10). Figure 2 shows the optimum configuration obtained by the continuous approximate optimization with a distortion ratio $g = 0.6595$. Since the optimum configuration is not realizable, nearby feasible positions were tried and the best one (with $g = 0.7028$) is also shown in Fig. 2. That position was analyzed using the exact analysis and g was found to be 0.7247. This represents an error of only 3% due to the modal approximation. Then the ESPS method was started from that design and the ribbed configuration in Fig. 3 was obtained with $g = 0.69190$. Various other starting points were tried and they all resulted in designs with virtually the same g. Two of those designs are shown in Fig. 3.

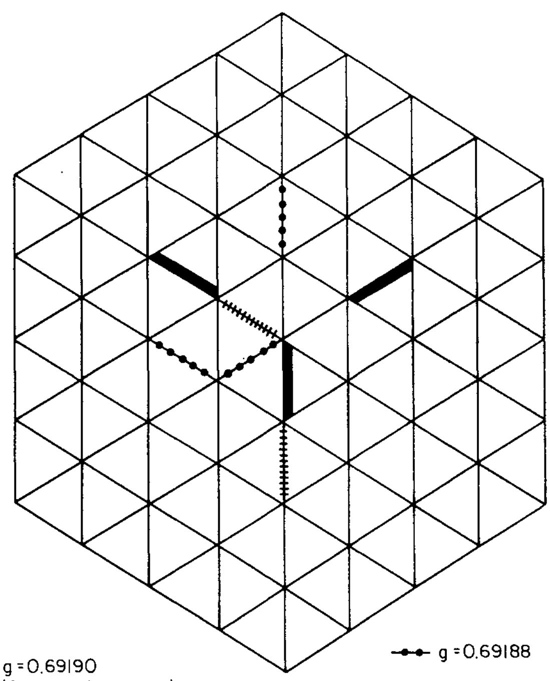

Fig. 3 Optimum locations of three actuators by discrete (ESPS) approach using different initial designs.

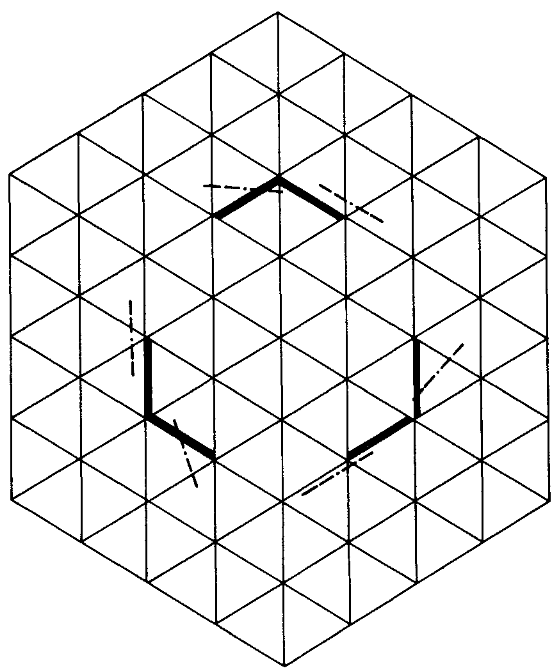

Fig. 4 Optimum locations of six actuators by continuous approximate optimization.

Table 1 Variation of distortion ratio g for best ESPS designs with increasing number of actuators

Number of actuators	3	4	5	6	7	8	9	10	11
Distortion ratio g	0.6919	0.6586	0.6253	0.5967	0.5704	0.5457	0.5271	0.5088	0.4895
Number of actuators	12	13	14	15	16	17	18	19	20
Distortion ratio g	0.4710	0.4443	0.4255	0.4061	0.3847	0.3710	0.3574	0.3450	0.3343

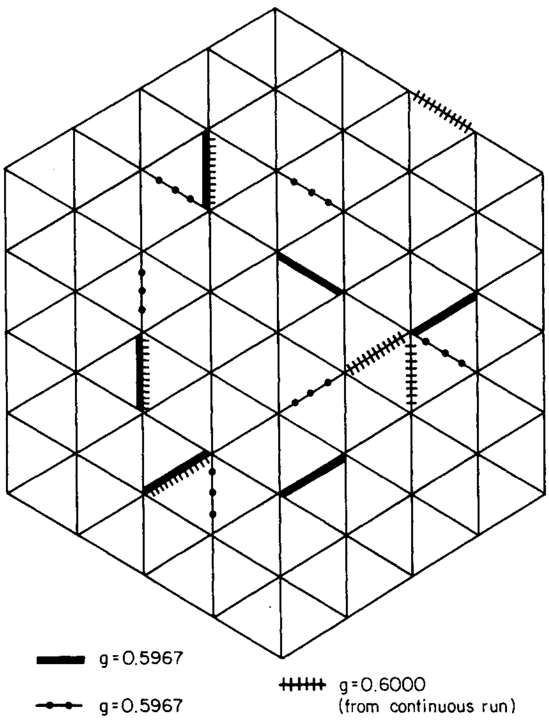

Fig. 5 Optimum locations of six actuators by discrete (ESPS) approach using different starting designs.

The same procedure was repeated for a case with six actuators. Figure 4 shows the optimum configuration obtained by the continuous approximate optimization with $g = 0.5510$. The best near-feasible design gave a value of $g = 0.6039$. The exact analysis gave a value of $g = 0.6358$ for that design indicating an error of about 5% for the approximate analysis. The ESPS method was then started from here and obtained the ribbed designed in Fig. 5 with $g = 0.6000$. Several other starting points were tried, and the ESPS method converged to designs with distortion ratios varying between 0.5967 and 0.6035. Two of the best designs are shown in Fig. 5.

The low scatter in the objective function g observed for the ESPS designs obtained from different initial designs, and the good agreement with the approximate optimization approach lead us to believe that the ESPS method provided good optimal, or near optimal designs.

Next, the number of actuators was increased one at a time from six to twenty. The optimization was performed using the ESPS method because the cost of the continuous optimization became very high as the number of actuators was increased (most of the computation cost was associated with the interpolation of the modes). To improve the chance of obtaining good designs the starting configuration for ESPS with n actuators included the n-1 optimum locations obtained for the n-1 actuator case. The results are summarized in Table 1. It is seen that the distortion ratio decreases very slowly with increasing number of actuators reaching a value of 0.33 with twenty. This indicates that for large reductions in manufacturing errors many more actuators are required. On the other hand, with only 61 nodes on the upper surface that influence the error, we know that 61 actuators would provide for complete elimination of the error. It appears that for large reductions of manufacturing errors the number of actuators must be close to 61. It is possible, however, that by locating actuators in core elements the number of required actuators could be reduced.

Concluding Remarks

The problem of efficient analysis of the statistics of initial and corrected shape distortions in space structures was addressed. Two approaches for improving efficiency were considered. One was an adjoint technique for calculating distortion shapes. The other was a modal expansion in terms of pseudovibration modes.

The two techniques were applied to the problem of otimizing actuator locations on a 55-m radiometer antenna. The adjoint analysis technique was coupled with a discrete-variable optimization method. The modal approximation techique was coupled with a standard conjugate-gradient optimization method. The agreement between the two sets of results was found to be good, validating both the approximate analysis and the optimality of the results.

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